Geodesics on $\mathrm{GL}(3)$ and nonlinear elasticity

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For a constitutive law in nonlinear elasticity theory, we consider the well-known *quadratic Hencky strain energy*

$$W(F) = \mu\|\operatorname{dev}\log U\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log U)]^2 = \mu\|\operatorname{dev}\log\sqrt{F^TF}\|^2 + \frac{\kappa}{2}[\operatorname{tr}(\log\sqrt{F^TF})]^2$$

of a deformation gradient $F \in \mathrm{GL}^+(3)$, where $\|\,.\,\|$ denotes the Frobenius matrix norm, tr is the trace operator and $\operatorname{dev} M$ denotes the deviator of a matrix M. We show that this energy function can be characterized as the *geodesic distance*

$$\operatorname{dist}^2_{\mathrm{geod}}(F, \mathrm{SO}(3)) = \inf_{Q\in\mathrm{SO}(3)} \operatorname{dist}^2_{\mathrm{geod}}(F, Q) = W(F)$$

of F to the special orthogonal group SO(3) induced by a Riemannian metric on the general linear group GL(3) which is left-GL(3)-invariant as well as right-O(3)-invariant. The proof requires an explicit parametrization formula for geodesic curves on GL(3) discovered by A. Mielke as well as a novel optimality result by Neff et al. involving the matrix logarithm and the unitary polar factor of an invertible matrix. Furthermore, we will consider the question whether SO(3) is *geodesically convex* in $\mathrm{GL}^+(3)$ with respect to a left-GL(3)-invariant, right-O(3)-invariant metric.

Our results suggest that certain quantities depending on the principal logarithm of F, which appear as geodesic distances of F to SO(3), can be regarded as purely geometric measures of the strain of a deformation. We therefore introduce a new hyperelastic constitutive model based on an energy function which depends on these quantities alone, approximating the classical Hencky strain energy for small strains.