Anisotropy of higher order tensors

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An enhanced kinematics model

Extended kinematics

- $\varepsilon^{(2)} = \varepsilon^{(ij)}$ is the strain tensor;
- $\eta^{(3)} = \varepsilon^{(2)} \otimes \nabla = \varepsilon^{(ij),k}$ is the 1st strain gradient tensor;
- $\kappa^{(4)} = \varepsilon^{(2)} \otimes \nabla \otimes \nabla = \varepsilon^{(ij),(kl)}$ is the 2nd strain gradient tensor.

Hyperstress tensors

Therefore by duality

- $\sigma^{(2)} = \sigma^{(ij)}$ is the Cauchy stress tensor;
- $\tau^{(3)} = \tau^{(ij)k}$ is the 1st hyper stress tensor;
- $\omega^{(4)} = \omega^{(ij)(kl)}$ is the 2nd hyper stress tensor.

with (..) indicates index symmetry
Mindlin’s 2nd strain gradient constitutive law:

\[
\begin{align*}
\sigma^{(2)} &= E^{(4)} : \varepsilon^{(2)} + M^{(5)} : \eta^{(3)} + N^{(6)} : \kappa^{(4)}, \\
\tau^{(3)} &= M^{T(5)} : \varepsilon + A^{(6)} : \eta^{(3)} + O^{(7)} : \kappa^{(4)}, \\
\omega^{(4)} &= N^{T(6)} : \varepsilon^{(2)} + O^{T(7)} : \eta^{(3)} + B^{(8)} : \kappa^{(4)}
\end{align*}
\]
Constitutive tensors are of two types:

- **Proper**
  - $E^{(4)} = E_{(ij)}^{(lm)}$ is the elasticity tensor;
  - $A^{(6)} = A_{(ij)k}^{(lm)n}$ is the $1^{\text{st}}$ strain-gradient elasticity tensor;
  - $B^{(8)} = B_{(ij)(kl)}^{(mn)(op)}$ is the $2^{\text{nd}}$ strain gradient elasticity tensor.

- **Coupling**
  - $M^{(5)} = M_{(ij)(lm)n}$ is the $C-1^{\text{st}}$ gradient coupling tensor;
  - $O^{(7)} = O_{(ij)k(lm)(no)}$ is the $1^{\text{st}}-2^{\text{nd}}$ strain gradient coupling tensor;
  - $N^{(6)} = N_{(ij)(lm)(no)}$ is the $C-2^{\text{nd}}$ strain gradient coupling tensor.

with \_\_\_ the invariance with respect to the underlined blocks permutations.
For centro symmetric media

In such a case, odd-order tensors vanish:

\[
\begin{cases}
\sigma^{(2)} = E^{(4)} : \varepsilon^{(2)} + N^{(6)} :: \kappa^{(4)}, \\
\tau^{(3)} = A^{(6)} :: \eta^{(3)}, \\
\omega^{(4)} = N^{T(6)} : \varepsilon^{(2)} + B^{(8)} :: \kappa^{(4)}
\end{cases}
\]

Therefore, the behavior is defined by:

- 3 proper tensors: \( E_{(ij)(lm)}, A_{(ij)k(lm)n}, B_{(ij)(kl)(mn)(op)} \)
- 1 coupling tensor: \( N_{(ij)(lm)(no)} \)

**Remark**

Conversely to FSGE, SSGE is remains coupled for centro symmetric media.
Synthesis

Observations

Higher order elasticity theories are coupled behaviors involving new higher order tensors.

Some questions

- how many different anisotropic systems there exists?
- what are the anisotropic features of these theories?
- is there any periodic media transverse isotropic for this model?

These questions will be investigated in terms of constitutive tensor spaces.
Let $\mathbb{T}^{(n)} \subseteq \otimes^n \mathbb{R}^3$ be a tensor space. Some questions:

- In how many symmetry classes is $\mathbb{T}^{(n)}$ divided?
- What are the type of these anisotropic system?
- Is there any general rule behind those answers?

**Aims**

The aim of this talk will be to solve these questions for $n = 2p$;
Before introducing results, some definitions, answering the following questions, are needed:

**Symmetry group** What is the characterization of symmetry?

**Symmetry class** How to compare symmetric object modulo their orientations?

**SO(3)-subgroup** Is there a classification of allowed symmetry classes?
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   - The limitations of the Forte-Vianello approach

4. Some new results
   - General results for even-order tensors
   - Constitutive tensor spaces

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   - Synthesis and conclusion
Symmetry groups

**Definition**

For any element of $E$, the set of operations $g$ in $G$ letting this element invariant is defined as

$$\Sigma_x := \{ g \in G \mid g \cdot x = x \}$$

**Example**

For an elasticity tensor\(^a\):

$$\Sigma_{E_{(ij)(kl)}} := \{ g \in SO(3) \mid g \cdot E_{(ij)(kl)} = E_{(ij)(kl)} \}$$

\(^a\)For explicit computation, the action has to be specified.
Symmetry groups

Beware!

Symmetry groups are relative to an orientation! In the above figure the 3 symmetry groups are different (but conjugate).

The good notion

To compare symmetry group modulo their orientations, we need to introduce the concept of symmetry class.
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Symmetry classes (1)

The set of all the positions an object can have, is its $G$-orbit:

$$\text{Orb}(x) := \{ g \cdot x \mid g \in G \} \subset E$$

Elements on the same orbit have conjugate symmetry groups, i.e.

$$\text{Orb}(x) = \text{Orb}(y) \Rightarrow \exists g \in G \mid \Sigma_x = g \Sigma_y g^{-1}$$

Figure: The symmetry class of orbit’s elements is $[D_4]$
Definition

The conjugacy class of a subgroup $K \subset G$ by

$$[K] = \{K' \subset G | \exists g \in G, K' = gKg^{-1}\}$$

A symmetry class $[\Sigma]$ is defined as the conjugacy class of an isotropy subgroup $\Sigma$.

Practical consequence

- A symmetry group is attached to any $E$ elements;
- $E$ elements having conjugate symmetry groups are in the same class;
- this is indeed an equivalence relation, thus partitioning $E$. 
**Property**

For compact Lie group, the number of symmetry classes is finite [?]:

\[ \mathcal{I}(E) := \{ [e]; [\Sigma_1]; \cdots ; [\Sigma_l] \} \]

where \( \mathcal{I} \) gives the set of isotropy classes of \( E \) elements.

**Application to tensor spaces**

For tensor spaces in \( \mathbb{R}^3 \):

- \( E = \mathbb{T}^n \subset \otimes^n \mathbb{R}^3 \);
- In general \( G = O(3) \), but in case of \( E = \mathbb{T}^{2p} \), \( G = SO(3) \).

And elements of \( \mathcal{I}(\mathbb{T}^n) \) are conjugate to \( SO(3) \)-closed subgroups.
SO(3)-closed subgroups

Symmetry group of $2p$-order tensors are conjugate to SO(3)-closed subgroups [?, ?]:

**Lemma**

*Every closed subgroup of SO(3) is conjugate to precisely one group of the following list:*

$$\{\mathbb{1}, \mathbb{Z}_n, D_n, \text{SO}(2), \text{O}(2), \mathcal{T}, \mathcal{O}, \mathcal{I}, \text{SO}(3)\}$$

Among them, we can distinguish:

**planar groups** $\mathbb{1}, \mathbb{Z}_n, D_n, \text{SO}(2), \text{O}(2)$ which are O(2)-closed subgroups;

**exceptional groups** $\mathcal{T}, \mathcal{O}, \mathcal{I}$ which are symmetry groups of platonian polyhedrons.
Planar groups

- $Z_n \ (n \geq 2)$ the cyclic group of order $n$, generated by the $n$-fold rotation $Q(k = \frac{2\pi}{n})$, and $SO(2)$ the associated limit group;
- $D_n \ (n \geq 2)$ the dihedral group of order $2n$ generated by $Z_n$ and $Q(i; \pi)$, and $O(2)$ the associated limit group.

(a) $Z_3$, chiral figure  
(b) $D_3$, regular triangle
Classes of exceptional subgroups are: $\mathcal{T}$ the tetrahedral group, $\mathcal{O}$ the octahedral group and $\mathcal{I}$ the icosohedral one:
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References
2\textsuperscript{nd} order symmetric tensor

Using the spectral theorem in finite dimension the result is direct.

\textbf{Result}

Let $T_{(ij)}$ be a symmetric 2\textsuperscript{nd} order tensor, then $T_{(ij)}$ is either: orthotropic ($[D_2]$); transverse isotropic ($[O(2)]$) or isotropic ($[SO(3)]$).

\begin{itemize}
  \item $[D_2]$
  \item $[O(2)]$
  \item $[SO(3)]$
\end{itemize}

\textbf{But...}

For higher order tensors, the determination of the set of symmetry classes is no more direct.
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There exists at least 5 different approaches:

**Irreducible decomposition [?]** : Lengthy but general and trustworthy;

**Symmetry planes [?, ?]** : In general, this method underestimates the number of symmetry classes;

**Dimension of invariant spaces** : In general, this information is incomplete to conclude.

**Complex action [?]** : This method is rather ad-hoc and was just applied to elasticity;

**Spectral decomposition [?]** : Very tricky too use, and not suited to coupling tensors.
Elasticity

Let $\mathbb{E}_{la}$ be the space of elasticity tensors

$$\mathbb{E}_{la} = \{ \mathbf{E} \in \mathbb{G}^4 | E_{ij} (kl) \}$$

Forte and Vianello [?] proved the following theorem

**Theorem**

$\mathbb{E}_{la}$ is divided into the following 8 symmetry classes:

$$[\mathbb{E}_{la}] = \{ [1], [Z_2], [D_2], [D_3], [D_4], [O(2)], [O], [SO(3)] \}$$

i.e. triclinic, monoclinic, orthotropic, trigonal, tetragonal, transverse isotropic, cubic and isotropic.
Using Forte-Vianello approach, some other results were obtained:

<table>
<thead>
<tr>
<th>Property</th>
<th>Tensor</th>
<th># classes</th>
<th>Action</th>
<th>Studied in</th>
</tr>
</thead>
<tbody>
<tr>
<td>Photoelelasticity</td>
<td>$T_{(ij)(kl)}$</td>
<td>12</td>
<td>$SO(3)$</td>
<td>[?]</td>
</tr>
<tr>
<td>Piezoelectricity</td>
<td>$T_{(ij)k}$</td>
<td>15</td>
<td>$O(3)$</td>
<td>[?]</td>
</tr>
<tr>
<td>Flexoelectricity</td>
<td>$T_{(ij)kl}$</td>
<td>12</td>
<td>$SO(3)$</td>
<td>[?]</td>
</tr>
<tr>
<td>6-th order tensors</td>
<td>$T_{ijklmn}$</td>
<td>14 or 17</td>
<td>$SO(3)$</td>
<td>[?]</td>
</tr>
</tbody>
</table>

Let's show the different symmetry classes of the piezo/flexo tensor spaces [?].
Toward a general result?

An observation

The number of symmetry classes for 4-th order tensors seems to be either 8 or 12. Exhaustive study of 6-th order tensors leads to either 14 or 17.

\[\mathbb{E}la = \{[1], [Z_2], [D_2], [D_3], [D_4], [O(2)], [\mathcal{O}], [SO(3)]\}\]

\[\mathbb{P}ho = \mathbb{F}lex = \{[1], [Z_2], [D_2], [Z_3], [D_3], \ldots, [Z_4], [D_4], [SO(2)], [O(2)], [\mathcal{T}], [\mathcal{O}], [SO(3)]\}\]

A question

Is there any rule behind this observation?
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The Forte and Vianello method is general, but suffers at least two drawbacks:

1. The explicit computation of the harmonic decomposition;
   - Cumbersome for 4-th order tensors;
   - Merely intractable for higher order tensors.

2. The specificity of the study for each kind of tensor.
   - Not suited for obtaining general results.

⇒ A new method has to be developed
With M. Olive (PhD student) some general theorems have been obtained, based on the following approach

**Geometric analysis** Complete characterization of intersection of $\text{SO}(3)$-subgroups classes, have been studied;

**Irreducible components** Results are then translated into intersection properties of symmetry classes of harmonic tensors;

**Constitutive tensors** As harmonic tensors are building blocks of general representations, results are extended to constitutive tensors.
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Let $G^{2p} = \otimes^{2p}(\mathbb{R}^3)$ be the space of tensors with no index symmetry;

**Lemma**

The symmetry classes of $G^{2p}$ are:

\[
\mathcal{I}(G^{2}) = \{ [1], [Z_2], [D_2], [SO(2)], [O(2)], [SO(3)] \}
\]
\[
\mathcal{I}(G^{4}) = \{ [1], [Z_2], \cdots, [Z_4], [D_2], \cdots, [D_4], [SO(2)], [O(2)], [\mathcal{T}], [\mathcal{O}], [SO(3)] \}
\]
\[
p \geq 3, \quad \mathcal{I}(G^{p}) = \{ [1], [Z_2], \cdots, [Z_{2p}], [D_2], \cdots, [D_{2p}], [SO(2)], [O(2)], [\mathcal{T}], [\mathcal{O}], [\mathcal{T}], [SO(3)] \}
\]
Let $S^{2p} = S^{2p}(\mathbb{R}^3)$ be the space of completely symmetric $2p$-th order tensors;

**Lemma**

The symmetry classes of $S^{2p}$ are:

$$
\mathcal{I}(S^2) = \{[D_2], [O(2)], [SO(3)]\}
$$

$$
\mathcal{I}(S^4) = \{[1], [Z_2], [D_2], [D_3], [D_4], [O(2)], [\mathcal{O}], [SO(3)]\}
$$

$$
p \geq 3, \quad \mathcal{I}(S^{2p}) = \{[1], [Z_2], \ldots, [Z_{2(p-1)}], [D_2], \ldots, [D_{2p}], [O(2)], [\mathcal{T}], [\mathcal{O}], [\mathcal{I}], [SO(3)]\}
$$

**The difference**

For $p \geq 3$; the classes $[Z_{2p-1}], [Z_{2p}]$ and $[SO(2)]$ miss in $\mathcal{I}(S^{2p})$. 
Theorem

Let $\mathbb{T}^{2p}$ a space of tensors, then either $\mathcal{J}(\mathbb{T}^{2p}) = \mathcal{J}(\mathbb{S}^{2p})$ or $\mathcal{J}(\mathbb{T}^{2p}) = \mathcal{J}(\mathbb{G}^{2p})$.

As a corollary we directly have:

\[
\begin{array}{c|c|c|c}
 n & 1 & 2 & \geq 3 \\
 \#\mathcal{J}(\mathbb{S}^{2n}) & 3 & 8 & 2(2n + 1) \\
 \#\mathcal{J}(\mathbb{G}^{2n}) & 6 & 12 & 4n + 5 \\
\end{array}
\]

Remark

This explain the first observation that the number of class is either minimal or maximal.
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Construction

- Linear constitutive laws are linear applications between tensors spaces called *state tensor spaces* (STS in the following).
- STS are the primitive notion from which the CTS will be constructed.

<table>
<thead>
<tr>
<th>Physical notions</th>
<th>Mathematical space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient of primal variable</td>
<td>$\epsilon_1 \in E_1 \subseteq \otimes^p \mathbb{R}^3$</td>
</tr>
<tr>
<td>Fluxes of dual variable</td>
<td>$\varsigma_2 \in E_2 \subseteq \otimes^q \mathbb{R}^3$</td>
</tr>
<tr>
<td>Linear constitutive law</td>
<td>$C \in \mathcal{L}(E_1, E_2) \sim E_1 \otimes E_2$</td>
</tr>
</tbody>
</table>

**Table:** Physical and mathematical links
Construction

We therefore have

- for coupling tensors:
  \[ \mathcal{L}(E_1, E_2) \simeq E_1 \otimes E_2 \subset T^p \otimes T^q = T^{p+q=2n} \]

- for proper tensors:
  \[ \mathcal{L}^S(E, E) \simeq E \otimes^S E \subset T^{2p} \]

Examples of such constructions are provided in the following table:

<table>
<thead>
<tr>
<th>Property</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$\otimes$</th>
<th># classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elasticity</td>
<td>$T_{(ij)}$</td>
<td>$T_{(ij)}$</td>
<td>Symmetric</td>
<td>8</td>
</tr>
<tr>
<td>Photoelelasticity</td>
<td>$T_{(ij)}$</td>
<td>$T_{(ij)}$</td>
<td>Standard</td>
<td>12</td>
</tr>
<tr>
<td>Flexoelectricity</td>
<td>$T_{(ij)k}$</td>
<td>$T_i$</td>
<td>Standard</td>
<td>12</td>
</tr>
<tr>
<td>1st-gradient elasticity</td>
<td>$T_{(ij)k}$</td>
<td>$T_{(ij)k}$</td>
<td>Symmetric</td>
<td>17</td>
</tr>
</tbody>
</table>
For proper tensors

**Theorem**

Let consider \( \mathbb{T}^{2p} \), the space of tensors of a proper physics described by the tensor vector space \( E = \mathbb{T}^p \). If \( p \geq 3 \), and \( E \) solely defined in terms of its index symmetries, then \( \mathcal{I}(\mathbb{T}^{2p}) = \mathcal{I}(\mathbb{G}^{2p}) \).

**Remark**

For \( p = 1 \) the space of symmetric 2\(^{nd}\) order tensors is obtained;  
For \( p = 2 \) in the case of \( \mathbb{T}_{(ij)} \), the space of elasticity tensors is obtained.  

In these cases the number of classes is minimal, these are the only cases. The space of elasticity tensors is exceptional.
Theorem

Let consider $\mathbb{T}^{2p}$ the space of coupling tensors between two physics described respectively by two tensors vector spaces $E_1$ and $E_2$. If these tensor spaces are of orders greater or equal to 1, then $\mathcal{I}(\mathbb{T}^{2p}) = \mathcal{I}(\mathbb{G}^{2p})$. 
For centro-symmetric media, odd-order tensors vanishes:

\[
\begin{align*}
\sigma^{(2)} &= E^{(4)} : \varepsilon^{(2)} + N^{(6)} :: \kappa^{(4)}, \\
\tau^{(3)} &= A^{(6)} :: \eta^{(3)}, \\
\omega^{(4)} &= N^{T(6)} : \varepsilon^{(2)} + B^{(8)} :: \kappa^{(4)}
\end{align*}
\]

Therefore the behavior is defined by:

- 3 proper tensors: \(E_{ij}^{(lm)}, A_{ijk}^{(lm)n}, B_{ijkl}^{(mn)(op)}\)
- 1 coupling tensor: \(N_{ij}^{(lm)(no)}\)
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Let $\mathbb{F}_{\text{gr}}$ be the space of 1st strain-gradient elasticity:

$$\mathbb{F}_{\text{gr}} = \{ \mathbf{A}^{(6)} \in \mathbb{A}^6 | A_{(ij)k} (lm)n \}$$

A direct application of our theorem gives:

$$\mathcal{I}(\mathbb{F}_{\text{gr}}) = \{ [1], [Z_2], \cdots, [Z_6], [D_2], \cdots, [D_6],$$

$$[\text{SO}(2)], [\text{O}(2)], [\mathcal{T}], [\mathcal{O}], [\mathcal{Z}], [\text{SO}(3)] \}$$

$\mathbb{F}_{\text{gr}}$ is divided into 17 symmetry classes.
Let $\mathcal{S}_{\text{gr}}$ the space of 2\textsuperscript{nd} strain-gradient elasticity tensors:

$$\mathcal{S}_{\text{gr}} = \{ B^{(8)} \in \mathbb{B}^8 | B_{(ij)(kl)} (mn)(op) \}$$

A direct application of our theorem gives:

$$\mathcal{I}(\mathcal{S}_{\text{gr}}) = \{ [1], [Z_2], \cdots , [Z_8], [D_2], \cdots , [D_8], [SO(2)], [O(2)], [T], [O], [I], [SO(3)] \}$$

$\mathcal{S}_{\text{gr}}$ is divided into 21 symmetry classes.
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The C-$2^\text{nd}$ strain gradient coupling tensor.

Let $\mathcal{C}_{\text{es}}$ be the space of coupling tensors between classical elasticity and $2^\text{nd}$ strain-gradient elasticity:

$$\mathcal{C}_{\text{es}} = \{\mathbf{N}^{(6)} \in \mathbb{G}^6 | N_{ij}(kl)(mn)\}$$

A direct application of our theorem gives:

$$\mathcal{I}(\mathcal{C}_{\text{es}}) = \{[1], [Z_2], \cdots, [Z_6], [D_2], \cdots, [D_6],$$

$$[\text{SO}(2)], [\text{O}(2)], [\mathcal{T}], [\mathcal{O}], [\mathcal{I}], [\text{SO}(3)]\}$$

$\mathcal{C}_{\text{es}}$ is divided into 17 symmetry classes.
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Chirality

A general result

Higher order elasticity is chiral dependent.

Figure: First strain elasticity tensor in 2D (constructed by homogenization [?])
Isotropy

A general result

Due to crystallographic restriction in $\mathbb{R}^3$, periodic media are neither isotropic nor transverse isotropic for higher order elasticity.

$D_8$-invariance

Transverse hemi/isotropy

Such a behavior may appear in quasi-crystals.
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To sum up

- For $2p$-order tensors, general results have been obtained, solving the associated classification problem;
- The space of elasticity tensors is exceptional;
- Higher order elasticities are chiral dependent behaviors;
- Transverse hemi/isotropy can only be produced in quasi periodic media.
Some extensions are needed to completely solved the problem:

- Extend the present result to $2p + 1$ order tensor spaces
- Combine all these results to formulate a complete conclusion on the constitutive law.

The first point is the object of a publication with M. Olive (nearly finished). Some prior remark concerning $2p + 1$ order tensors:

- The groups to consider is $O(3)$, conjugacy classes are closed $O(3)$ subgroups;
- The number of symmetry classes is higher that for even order tensor.
- $\#I(M^{(5)}) = 29$ and $\#I(O^{(7)}) = 40$. 
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*Introduction to compact transformation groups.*  

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