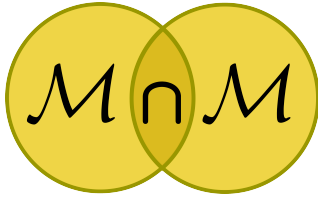


UNIVERSITÀ DEGLI STUDI DELL'AQUILA

M&MOCS

*International Research Center on*  
**MATHEMATICS AND MECHANICS**  
**OF COMPLEX SYSTEMS**



M&MoCS Workshop on “**SECOND GRADIENT and GENERALIZED CONTINUA**”  
**12-16 March 2012** at the “Palazzo Caetani” in Cisterna di Latina, Italy

organised by Francesco dell'Isola and Samuel Forest  
International Research Center M&MoCS <[memocs.secretariat@gmail.com](mailto:memocs.secretariat@gmail.com)>

**Scientific Program of the Workshop**

**Monday, March 12<sup>th</sup>**

**Opening - Session : General concepts for second gradient media I**

16:00-17:30 Francesco dell'Isola and Pierre Seppecher

**Tuesday, March 13<sup>rd</sup>**

**09:00-12:00: Session : General concepts for second gradient media II**

09:00-10:15 Nicolas Auffray

10:15-10:45 coffee-break

10:45-12:00 Martin Ostoja Starzewski

**14:00-17:30 : Session : Generalised continua and homogenization I**

14:00-15:00 Claude Boutin

15:00-16:00 Antonio Carcaterra

16:00-16:30 coffee-break

16:30-17:30 Samuel Forest

**Wednesday, March 14<sup>th</sup>**

**09:00-12:30: Session : Generalised continua and homogenization II**

09:00-10:00 Jia Li

10:00-11:00 Karam Sab

11:00-11:30 coffee-break

11:30-12:30 Anil Misra

**14:00-17:30: Session: Generalised continua and homogenization III**

14:00-15:00 Harm Askes

15:00-16:00 Holger Steeb

16:00-16:30 coffee-break

16:30-17:30 Angela Madeo

18:30 Visit to Sermoneta and dinner

**Thursday, March 15<sup>th</sup>**

**09:00-12:30 Session : Dynamics**

09:00-10:00 Stefan Diebels

10:00-11:00 Ralf Jaenicke

11:00-11:30 coffee-break

11:30-12:30 Luca Placidi

**14:00-17:30 Session : Beams and plates**

14:00-15:00 Holm Altenbach

15:00-16:00 Noël Challamel

16:00-16:30 coffee-break

16:30-17:30 Victor Eremeyev

**Friday, March 16<sup>th</sup>**

**09:00-11:30 Session : Strain localization phenomena**

09:00-10:00 Denis Caillerie

10:00-10:30 coffee-break

10:30-11:30 Samuel Forest

## **Abstracts**

### **Gradient Elasticity in Dynamics**

#### **Harm Askes**

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The use of gradient elasticity is often associated with static boundary value problems, where the higher-order gradient terms are used to remove the stress and/or strain singularities from crack tip fields. However, gradient elasticity is also a suitable modelling strategy in dynamics, whereby the primary aim is to capture dispersive wave propagation. The theories of Mindlin, Eringen and Aifantis will be presented and compared, together with their finite element discretisations. Next, wave dispersion through heterogeneous materials will be used to steer the identification and quantification of the length scale parameters of gradient elasticity - either via experiments, upscaling techniques or homogenisation methods.

## **Anisotropy of higher-orders gradient elasticity**

**Nicolas Auffray**

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The aim of this talk is to discussed on the determination and the representation of the higher-order elasticity tensors of Mindlin first and second strain-gradient theory.

The first rigorous proof that the space of elasticity tensors is divided into 8 symmetry classes was made in 1996 by Forte and Vianello. Since then, this method has been applied to other constitutive laws. But, despite of being effective the method suffer the drawbacks to be cumbersome to applied and to not provide general results. Furthermore, its complexity increases with the order of the considered tensor space [2,6].

Recently, we introduced a new geometric approach of the question. This method gives a direct and complete answer for even-order tensors, no matter their orders [1]. As an application, we obtain the number and the type of symmetry classes of the two 6<sup>th</sup>-order tensors and the 8<sup>th</sup>-order tensor of Mindlin second strain gradient elasticity.

In a second part of the talk, the matrix representation of the 6<sup>th</sup>-order first gradient elasticity tensor will be discussed, both for 2D [4, 5] and 3D spaces [3]. The shape of these matrices, in certain anisotropy systems, shows that higher-order elasticity is chiral-sensitive.

[1] Marc Olive, Nicolas Auffray, Symmetry classes for even-order tensors, to be submitted ;

[2] Le Quang H., Auffray N., He Q.-C., Bonnet G., 2012. Symmetry groups and symmetry classes for sixth-order tensors. to be submitted ;

[3] Auffray N., Le Quang H., He Q.-C., Bonnet G., 2012. Matrix representation for 3-dimensional anisotropic strain-gradient elasticity. to be submitted ;

[4] Nicolas Auffray, Régis Bouchet, Yves Bréchet, Strain gradient elastic homogenization of a bidimensional cellular media, *International Journal of Solids and Structures* (2010) 47,1698-1710.

[5] Nicolas Auffray, Régis Bouchet, Yves Bréchet, Derivation of anisotropic matrix for bi-dimensional strain gradient elasticity, *International Journal of Solids and Structures* (2009) 46, 440-454.

[6] Nicolas Auffray , Décomposition harmonique des tenseurs -Méthode spectrale-, *Comptes Rendus Mécanique* (2008), 336, 370-375.

## **Design of space / time non-local elastic composite through homogenization**

**Claude Boutin<sup>1</sup>, S. Hans<sup>1</sup>, C. Chesnais<sup>2</sup>, J. Soubestre<sup>1</sup>**

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This paper deals with the design of elastic composites behaving as generalized media with non-local effects in space and/or time at the leading order. This issue requires to relate the physics at the scale of the internal microstructure to the macroscopic behaviour. Hence, the theories of generalized continua postulated at the macroscale, e.g. [1], are not appropriate for this question. Conversely, the homogenization method of periodic media, combined with a systematic use of scaling is an efficient approach for our purpose. The present study is developed in this framework, based on the key assumption of micro-macro scale separation.

It has been established by high order homogenisation [2], that elastic composites present non-local effects in space and time. However, this latter effect appears as correctors of the classical Cauchy behaviour at the leading order. Therefore, to enhance the non local effects up to the leading order, the physics at the local scale must be modified, either by introducing a contrast in the “geometry” of the constituents (instead of  $O(1)$  aspect ratio as in [2]), or by increasing the contrast of elastic properties (taken  $O(1)$  in [2]). These options are analysed by considering reticulated materials or highly contrasted composites.

Reticulated materials that present an high contrast between shear and compression deformability are investigated through the homogenization method of discrete media. The conditions of second gradient behaviour (non local in space at the leading order) are identified according to the parameters of the cell and the dimensions of the shearing zone [4]. Further, weakening the 3D-periodicity condition to 1D periodicity enables to obtain Cosserat description related to generalized beam behaviours. In dynamics, inner resonance may also arise and induces a non-Newtonian description. As in meta-materials, this non locality in time leads to dispersive P waves, with cut-off frequencies [5].

Similar results are established for elastic composites made of a soft matrix periodically reinforced by linear slender inclusions of high shear modulus. One obtains a second gradient continuum accounting for the inner bending of inclusions and the shear behaviour of the matrix [3] (experiments on specimens designed following these theoretical studies confirm this description). In accordance with a non local spatial effect at the leading order, such materials present an internal intrinsic length much larger than the period size. In dynamics, the strong contrast induces a phenomenon of inner resonance of the soft matrix. This results in an apparent mass depending on the frequency, i.e. a non-Newtonian description corresponding to “meta-materials”.

To conclude, both analyses show that a key ingredient to reach a generalized behaviour at the leading order is the large contrasts of mechanical properties within the microstructure. In appropriate loading conditions, it enables enriched local kinematics (global rotation and/or inner deformation i.e. non locality in space) but also inner resonance (non locality in time).

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## **Second grade modeling as a regularization of the strain localization and related topics**

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Strain localization is a rather common phenomenon in non elastic (for instance elasto-plastic or damageable) materials. Particularly in geomechanics, it can be a fracture precursor or be at the origin of a failure so it is important to be able to model it. From a mathematical point of view, the strain localization can be seen as a bifurcation in the quasi static evolution of non elastic material. In the Cauchy's theory of continuous media, some criteria, as Rice's one, can be used to detect the onset of a localization but, unfortunately, the modeling of the post localization evolution does not seem possible in that framework. It may indeed be shown, as well in numerical FE simulation as with theoretical closed form solutions, that the problem presents infinitely many solutions. Moreover there is no internal length in a Cauchy's modeling which might give a shear band thickness. The post localization modeling needs some kind of regularization, those which have been proposed are: dynamics and viscosity or generalized continuum theories such as non local, micromorphic, Cosserat's or second grade theories. At Grenoble 3SR laboratory, the regularization that was chosen some fifteen years ago, is the second grade continuum theory in the spirit of the works by Mindlin and Germain, that is to say by considering continuum kinematics enriched with the gradient of the strain in preference to considering space gradient of parameters, as plastic strain, which mathematically and numerically seems less sound. In that talk, will be presented some theoretical and numerical developments about the regularization effect of the use of the second grade continuum theory on the modelling of strain localization such as closed form solution for softening behaviour, full second grade elasto plasticity, simplified second gradient modeling for dilatant materials (work by Fernandes, Chavant and Chambon), second gradient for damageable concrete (work by Kotronis and Chambon) and other related topics.

## **Second gradient, nonlocal and quantum effects in nanostructures**

Prof. **Antonio Carcaterra**

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In recent times in structural dynamics a new challenging field is open. The chance of controlling mechanical systems at the smallest scales, implies to reconsider the related models. Granular effects become important, and the chance to use continuous models to approach discrete set of interacting particles compels to introduce novel constitutive relationship. Additionally, the small scale combined with extremely high-frequency and low energies in nanosystems, forces the inclusion of quantistic effects. This paper is aimed at exploring the use of second gradient theories, linear and nonlinear, to produce a continuous model accounting for microscale granular ploidatomic interactions and nonclassical effects.

## **Buckling and vibrations of higher-order shear gradient and nonlocal beams**

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This paper is devoted to the buckling and vibration behavior of generic higher-order shear beam models, investigated in a unified framework. We show that most higher-order shear beam models developed in the literature (polynomial, sinusoidal, exponential shear strain distribution assumptions over the cross section) can be classified in a common gradient elasticity Timoshenko theory, whatever the shear strain distribution assumptions over the cross section. The governing equations of the buckling/vibrations problem are obtained from a variational approach leading to a generic sixth-order differential equation with variationally consistent boundary conditions. The model valid for macrostructures is generalized for micro or nanostructures using some nonlocal theories to account for small scale effects. We both use the Eringen's based integral theory and the gradient theory to derive the buckling and vibration differential equations. The nonlocal framework is also developed in a variational consistent framework. Buckling and vibrations solutions are presented for the nonlocal higher-order beam/column models with pinned-pinned boundary conditions. We finally discuss the main characteristics of both nonlocal theories to capture the small scale effects for micro and nanostructures.

Keywords: Higher-order shear beam; Gradient elasticity Timoshenko; Variational approach; Buckling; Nonlocal beam; Composite structures; nanostructures



## **Micro- and macro-mechanical modeling of foams and identification of Cosserat parameters**

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Foams are often used in light weight construction, therefore the modeling of foams is of increasing interest. A main problem is that foams show distinct size effects. In the present work, on one hand, a micromoechanical model of a polymeric foam based on data from computer tomography is used to replace real experimental data. On the other hand, a Cosserat approach is used to describe the behavior of the foam on a macro scale. Inverse analysis is used to identify the Cosserat parameters with respect to the data obtained form the microscopic model.

## **On the plates and shells theories with microstructure**

(with **Holm Altenbach**)

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We discuss the derivation of 2D shell equations made of material with microstructure. The following models of materials are considered:

1. Micropolar materials,
2. Micromorphic materials,
3. Materials of second-grade.
4. Materials with continuously distributed dislocations.

The main goal of the lecture is to investigate how 2D equations of plates and shells inherit the generalised properties of 3D continuum.

## **Generalized continua and non-homogeneous boundary conditions in homogenisation methods**

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Extensions of classical homogenization methods are presented that are used to replace a composite material by an effective generalized continuum model. Homogeneous equivalent second gradient and micromorphic models are considered, establishing links between the macroscopic generalized stress and strain measures and the fields of displacement, strain and stress inside a volume element of composite material. Recently proposed non-homogeneous boundary conditions to be applied to the unit cell, are critically reviewed [1]. In particular, it is shown that such polynomial expansions of the local displacement field must be complemented by a generally non-periodic fluctuation field. A computational strategy is introduced to unambiguously determine this fluctuation.

The approach is well-suited for elastic as well as elastoplastic composites [2].

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## **Strain gradient plasticity: Application to Lüders band propagation**

**A. Marais, M. Mazière, S. Forest**

Strain ageing constitutive models are suitable to simulate the formation and propagation of Lüders bands in complex specimens and components [1]. The identification of the corresponding material parameters is difficult because the strain localization phenomena associated with the Lüders behavior must be taken into account.

The spurious mesh dependence of standard finite element simulations of Lüders band propagation is illustrated in the present work and removed using a strain gradient plasticity model [2].

Furthermore the gradient approach introduces a characteristic size corresponding to the finite dimension of the Lüders band front as observed from strain field measurements.

The parameters are identified from the experimental measurement of the peak stress and plastic strain carried by the band for a C-Mn steel over a temperature range from 150°C to room temperature and for several strain rates.

The validity of the model is tested on 3D simulations of the Lüders band propagation in a strain gradient plasticity medium.

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[2] S. Forest, [Micromorphic approach for gradient elasticity, viscoplasticity and damage](#), ASCE Journal of Engineering Mechanics, vol. 135, pp. 117-131, 2009.

## **Wave propagation phenomena in micro heterogeneous materials by micromorphic continuum approaches**

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Micro heterogeneous materials feature highly complex material properties. Under quasi-static loading conditions, the overall material response is dominated by anisotropy and size dependent boundary layer effects on the one hand. On the other, high-porous cellular materials exhibit a strongly dispersive behaviour if exposed to acoustic waves.

In order to model such complex materials, we will discuss a homogenisation technique in terms of a mean-field approach. Doing so, we replace the micro heterogeneous material by a homogeneous counterpart accounting for identical (effective) material properties. However, standard (Cauchy) continuum methods are not suitable since they neither involve higher order effects such as boundary layers nor dissipative properties. Hence, we apply a micromorphic continuum formulation on the effective scale and pay special attention to the relations between the extended degrees of freedom and the underlying microscopical deformation states. The choice of representative unit cells will be highlighted as well as the appropriate formulation of loading conditions (affine, periodic, minimal).

The focal point of the present contribution will be the evaluation of the micromorphic equations of motions for exemplary microstructures as well as the comparison of the derived dispersion relations to results by discrete lattice simulations. Finally, we intend to discuss the overlap of discrete methods on the one hand and continuum formulations (Cauchy, Cosserat, micromorphic) on the other hand.

## **On the consistency of the strain gradient constitutive laws obtained from homogenization**

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Numerous authors believe that the homogenization on a RVE by including the gradient terms of the macroscopic field is a natural way to construct the high-order constitutive laws of heterogeneous materials. However, some fundamental aspects concerning the uniqueness and the consistency of the higher-order gradient laws thus obtained have not been thoroughly studied. For example, the higher-order effective moduli can depend on the size of the RVE. In the present work, we describe a homogenization method in order to provide a correction to the strain gradient constitutive relations such that this size dependence vanishes. The presented method includes two main steps. The first one is the construction of the average strain-energy density for a well-chosen Representative Volume Element (RVE) by using a homogenization technique. The second one is the transformation of the obtained average strain-energy density to that for the continuum. This transformation permits to ensure that the effective moduli don't depend on the size of a RVE, but only on the intrinsic size of the components. It also ensures that the moduli related to the strain gradient vanish as the contrast between the components disappears. These features were then illustrated by means of some simple examples. Numerical applications of this method were also discussed.

**The Effect of Fluid Flow and of Micro-Heterogeneities in Porous Media on Acoustic Compression Wave Propagation, Transmission and Reflection**

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In poromechanics a relevant role is played by coupling phenomena between compressible fluid flow and deformation of the solid matrix. The behavior of complex materials can be greatly influenced by such coupling phenomena. In this presentation, we discuss how confined streams of fluid in deformable porous matrices affect compression wave propagation and their reflection and transmission at a solid-material discontinuity surface.

Moreover, the possibility of accounting for some micro-heterogeneities in the solid matrix is taken into account by means of a second gradient theory. The effect on reflected and transmitted energy of the second gradient parameter is presented as well as the high-frequency behavior of the considered generalized continuum which shows significant differences from classical results obtained via Cauchy theory.

**Contact interactions in N-th gradient continua. Examples of microstructures with length scales leading to continua with these exotic interaction and applications to a problem of growing tissues**

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Cauchy format for continuum mechanics is not general enough to include the description of some important phenomena occurring in a large class of “microscopically” inhomogeneous bodies. This circumstance has been recognized already by E. and F. Cosserat and later by Mindlin, Rivlin, Toupin and Green. Particularly detailed is the development of second gradient model due to Paul Germain, which is the simplest continuum model which cannot be included in the Cauchy format.

Actually Cauchy Continuum model has a wide but limited scope of predictivity: therefore the logical possibility of generalizing it truly corresponds to the need of describing some new and interesting phenomena, which include, but are not limited to, the formation of many different kinds of boundary layers.

The class of generalized (N-th gradient) continua which is presented here is constituted by those models in which deformation energy may depend on  $N+1$  gradients of displacement field.

The precise definition of N-th gradient continua and the nature of contact interactions which may arise in them can be mathematically formulated and characterized only by using suitable concepts from the theory of distribution and differential geometry, but it is definitively not so difficult to generalize -in the considered context- the Cauchy concept of stress state.

The mathematical abstraction required does not seem inappropriate as it actually supplies a useful description of some phenomena of relevance in the mechanics of growing tissues and opens interesting perspectives in the design of artificial materials showing “exotic” mechanical behavior.

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- [3] F. dell'Isola, P. Seppecher and A. Madeo “How contact interactions may depend on the shape of Cauchy cuts in N-th gradient continua: approach “à la D'Alembert””, to appear in ZAMP – *Zeitschrift für angewandte Mathematik und Physik*

## **Micromechanics based second gradient continuum theory for cementitious granular materials**

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The mechanical behavior of cementitious granular materials is intimately linked to their microstructure and grain interactions. We have recently developed a micromechanical approach for modeling cohesive granular media by considering their underlying microstructure conceptualized as a collection of grains interacting through pseudo-bonds [1]. The advantage of this micromechanical approach is that the resultant continuum model retains the discrete nature by incorporating the effect of nearest neighbor grain interactions through the inter-granular force-displacement relationship and orientation vector. Furthermore, this micromechanical approach naturally leads to a 2<sup>nd</sup> gradient continuum theory for these materials involving strain gradient and its conjugated double stresses. Consequently, the internal length scale parameter in this model reflects the natural granularity of the underlying material microstructure. In this presentation, we will first describe the derivation of the constitutive relationships for this 2<sup>nd</sup> gradient theory. We will then utilize the method of virtual power, to develop the governing equations, boundary conditions and the weak form for this 2<sup>nd</sup> gradient theory [2, 3]. The applicability of the derived model will be demonstrated through examples with different imperfections and simulation of shear band failure. For numerical calculations, the weak form is discretized using an element-free Galerkin (EFG) formulation [4]. The model predictions are found to exhibit both quantitative and qualitative consistency with the observed behavior of cementitious granular material.

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## From Fractal Media to Continuum Mechanics

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While the book titles of [1,2] suggest that fractals are ubiquitous, the field of science that goes under the label of continuum mechanics seems to indicate that the physical matter is smooth. Of course, not everything in nature needs to be modeled by fractals, and if we encounter them, they always have the upper and lower cut-offs; these are the so-called pre-fractals. However, the challenge to adapt continuum models to fractal-type media is there. In this paper we begin with a review of works in this direction, dating back to 2005 [3,4] and following the dimensional regularization strategy in which the fractal dimension  $D$  of a fractal object is also the order of fractional integrals employed to state global balance laws. In effect, the governing equations are cast in forms involving conventional (integer-order) integrals, while the local forms are expressed through partial differential equations with derivatives of integer order but containing coefficients involving  $D$  and  $R$  (a resolution) as well as a surface fractal dimension  $d$  [5,6]. While the original formulation was based on a Riesz measure—and thus more suited to isotropic media—the newer model is based on a product measure capable of describing a local material anisotropy, i.e. direction-dependent fractal geometric properties of a given solid medium [7,8]. This measure generally results in the asymmetry of the Cauchy stress, which, in turn, leads to micropolar continuum mechanics [9,10]. Besides the development of equations of elasticity in static, dynamic, or finite strain settings [11], the theoretical formulation lends itself to numerical solutions of boundary value problems [12,13]. Next, given that the vector identities of vector calculus hold for our fractal gradient, divergence and curl operators, Maxwell's equations can be generalized to anisotropic fractals using either a variational principle [14] or, independently, via a conceptual approach (involving generalized Faraday and Ampère laws). Another application of the theory concerns fractally-structured fluid-type media, where we arrive at a micropolar continuum mechanics of fluent media [15].

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## **Linear elastic trusses leading to continua with exotic mechanical interactions**

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When studying the statics of a truss made of a discrete network of nodes joined by linear elastic springs, the kernel of the quadratic potential energy (space of floppy modes) plays an essential role. In particular when studying the asymptotics of the truss as the number of nodes becomes large : in the continuous homogenized description of the system one can recover the floppy modes. The fact that one does not commonly consider systems with finite dimension kernels other than the space of rigid motions is the reason why only displacements or rotations, or their dual counterparts, forces or torques, are transmitted by usual materials. We show how to design a truss in order to get an extra floppy mode which corresponds to a constant dilatation of the medium. Thus, forcing the dilatation in some part of the domain will tend to fix it on the whole domain. The mechanical interactions which are present in this continuum and responsible of this “transmission of dilatation” are not classical. We discuss and illustrate them using the discrete system of springs. We then give some examples of different possible floppy modes and the associated continuous models.

## **Stress gradient theories for the homogenization of materials and structures**

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A new plate theory for out-of-plane loaded thick plates where the static unknowns are those of the Kirchhoff-Love theory (3 in-plane stresses and 3 bending moments), to which six components are added representing the gradient of the bending moment, will be presented. The new theory, called the Bending-Gradient plate theory is an extension to periodic plates of the Reissner-Mindlin plate theory which appears as a special case when the plate is homogeneous.

Then, based on the same ideas, a new stress gradient continuum theory for the homogenization of periodic materials, that fundamentally differs from the well established strain gradient model, will be presented. It assumes that the deviatoric part of the gradient of the Cauchy stress tensor can contribute to the free energy density of solid materials. It requires the introduction of so-called micro-displacement degrees of freedom in addition to the usual displacement components.

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**Wave propagation through partially-saturated porous media:  
About extended continuum models for media with discontinuous phases**

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In the present contribution, we present a three-phase multi-scale model describing wave propagation phenomena in partially-saturated porous media. The effect of capillary forces on attenuation and phase velocities is investigated in the frequency range. On the one hand, the model is able to describe two continuous fluid phases, and, on the other hand the present approach models a continuous non-wetting phase and a discontinuous wetting phase. Furthermore, resonance effects of single liquid bridges and/or liquid clusters are captured with miscellaneous eigenfrequencies taking into account a viscoelastic restoring force (pinned oscillations and/or sliding motion of the contact line).

The aim of the current work is to develop and discuss a basic multi-scale modelling framework which takes into account the dynamics of statistically-distributed phases in a macro-scale continuum approach. It will be shown, that such a three-phase model allows to study frequency-dependent attenuation due to fluid oscillations and attenuation with respect to wave-induced flow. Furthermore, the distinct model could be applied to waves in reservoir rocks in the seismic range (field scale) and ultrasound range (lab scale). We will show that the results of the model are consistent with well-established limits, i.e. the biphasic poroelastic Biot model. In addition, we will show that for the quasi-static limit case the results of the model are identical with the phase velocity obtained by the Gassmann-Wood limit.

**Static and dynamic solutions in second gradient 3D continua: the case of Linear plane wave propagation and normal transmission and reflection at discontinuity surfaces with material properties**

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